Technical Note 22. Computing Percent Slope Gradient

22.0 Background

Slope, both its **gradient** and **aspect**, warrants particular attention because it governs water movement and energy inputs across the landscape. Slope gradient and aspect significantly influence soil development, hydrology, and erosion. Steeper gradients increase the velocity of surface and subsurface water flow, accelerating erosion and transporting dissolved nutrients downslope—creating fertility gradients with depleted summits and enriched toe slopes. Slope gradient is a key input to the Universal Soil Loss Equation and a critical parameter for erosion control design. Slope aspect affects soil and air temperature: in the Northern Hemisphere, south-facing slopes receive more direct solar radiation, resulting in higher temperatures, greater evaporation, lower soil moisture, and distinct vegetation communities compared to north-facing slopes.

22.1 Definition of Slope

The word 'slope' has many meanings. To the average person the word may convey the incline, or "steepness" or "pitch" of surfaces like a roof, wheelchair ramp, or pavement, relative to some invisible horizontal line. That intuition is essentially correct. However, to analyze slope we need a more precise definition. Fortunately, mathematics provides one.

Consider a two-dimensional Cartesian coordinate plane, the same one you learned about in grade school geometry (assuming you weren't passing notes during the lesson):

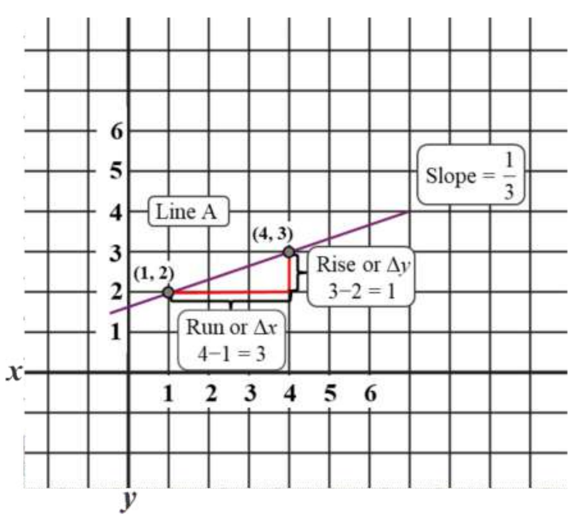


Figure 1. Two-dimensional Cartesian coordinate plane with x and y axes labeled.

Recall that for any two points on a straight line, the slope is calculated as:

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{3}$$

The "run" is the horizontal distance (Δx) between the two points represented by the ordered pairs (1, 2) and (4, 3). The "rise" is the vertical distance (Δy) between these two points. Slope is, therefore, the ratio of the vertical change to horizontal change between any two distinct points that indicates the steepness or incline of the line.

Slope can assume positive or negative values depending on the direction of the line. When the line is going up from left to right, the slope is positive. When the line is going down from left to right, the slope is negative. If a line is horizontal the slope is zero. If the line is vertical the slope is undefined.

It can be observed that connecting the line running through the ordered pairs (1, 2) and (4, 3) with the lines defining the vertical rise and horizontal run creates a right triangle:

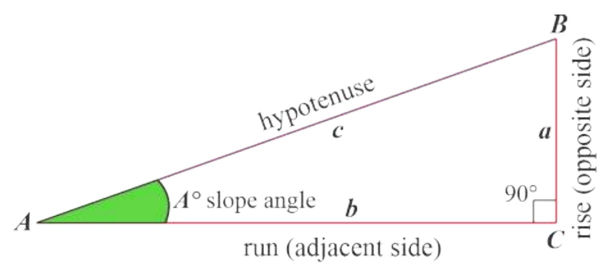


Figure 2. Right triangle with standard notation.

The notation in **Figure 2** substitutes letters $\boldsymbol{a}$ and $\boldsymbol{b}$ for the Cartesian quantities Δy and Δx, respectively. Slope angle $A°$ formed between side $\boldsymbol{c}$ (hypotenuse) and side $\boldsymbol{b}$ (adjacent) determines the steepness of the slope. Properties of a right triangle include:

- One of the three interior angles is 90 degrees.
- The longest side **c** is called the hypotenuse.
- Length of the three sides obey the Pythagorean rule.

The Pythagorean rule, better known as the **Pythagorean theorem,** applies to all right triangles:

$$\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ represents the three sides of a right triangle noted in **Figure 2**. This relation is extremely important in solving many problems involving slope, distance, and area.

Books on surveying often assign more descriptive names to the sides of a right triangle:

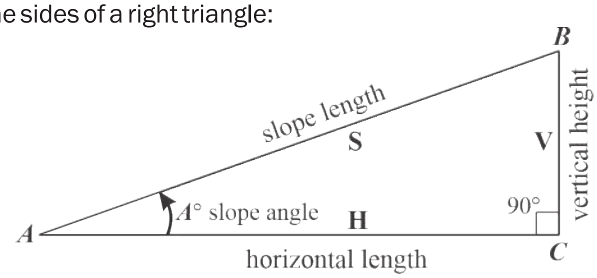


Figure 3. Right triangle with alternate notation.

Hereon, we'll use the **H**, **V**, **S** notation for convenience. Remember though, it doesn't matter how the sides are labeled so long as the Pythagorean rule is observed.

22.2 Slope, Grade, and Gradient

The terms slope, grade, and gradient are sometimes used indistinctly. In Euclidean geometry the concept of slope emphasizes, abstractly, the incline or steepness of a line with positive, zero, or negative value as described in Section 22.1. In the real world, **gradient** refers to the change in value of some physical quantity, such as temperature, pressure, or concentration, with respect to distance or time. In civil engineering and construction the term **slope gradient** describes the change in **elevation** over a given horizontal distance. **Grade** is also used to express vertical rise or fall over a given distance, usually as a percentage or ratio. In context of mathematics, gradient and grade express the *absolute value* of slope, i.e. relative distance from zero (horizontal) reference.

There are four ways to express slope gradient:

▶ **Percent**

$$\text{Gradient }(\%) = \frac{\text{rise }(\Delta y)}{\text{run }(\Delta x)} = \frac{\mathbf{V}}{\mathbf{H}} \times 100$$

▶ **Per mille**

$$\text{Gradient }(‰) = \frac{\text{rise }(\Delta y)}{\text{run }(\Delta x)} = \frac{\mathbf{V}}{\mathbf{H}} \times 1000$$

▶ **Angle**

$$\text{Gradient }(^\circ) = \arctan\left(\frac{\text{rise }(\Delta y)}{\text{run }(\Delta x)} = \frac{\mathbf{V}}{\mathbf{H}}\right)$$

▶ **Ratio**

$$\text{Gradient }(1{:}n) = \frac{1}{n} \text{ where } n = \frac{\text{rise }(\Delta y)}{\text{run }(\Delta x)} = \frac{\mathbf{V}}{\mathbf{H}}$$

Let's examine each of these in turn. First. notice that all expressions of slope gradient are given by the ratio of vertical rise to horizontal run, which true dirt hogs should recognize as "rise over run" or slope. Slope gradient expressed as a percentage is slope ×100. The expression "per mille" (denoted by the symbol ‰) is slope × 1000 interpreted as parts per thousand, for example meters (m) per kilometer (km). Slope angle, or the *angle of inclination*, is found using the arctangent, or *inverse tangent*[1] function with slope as the input. Slope gradient as a ratio uses the colon (:) notation in mathematics to express slope in fractional value "1 in *n*" or "1 to *n*" nomenclature.

Now, let's evaluate slope gradient for **V** = 1 m and **H** = 30 m, the length of a standard tape measure (~100').

$$\text{Gradient}(\%) = \frac{\mathbf{1}\text{ m}}{\mathbf{30}\text{ m}} \times 100 = 3.3\%$$

$$\text{Gradient }(‰) = \frac{\mathbf{1}\text{ m}}{\mathbf{30}\text{ m}} \times \frac{1000\text{ m}}{\text{km}} = \frac{33.3\text{ m}}{\text{km}} \text{ or } 33.3‰$$

$$\text{Gradient}(^\circ) = \arctan\left(\frac{\mathbf{1}\text{ m}}{\mathbf{30}\text{ m}}\right) = 1.9^\circ$$

$$\text{Gradient}(1{:}n) = \frac{1}{\frac{\mathbf{1}\text{ m}}{\mathbf{30}\text{ m}}} = 1 \times \frac{\mathbf{30}\text{ m}}{\mathbf{1}\text{ m}} = 1{:}30$$

[1] $\tan^{-1}$ button on a scientific/engineering calculator.

Stating slope gradient as 3.3% means that for every 30 m of horizontal distance traveled, elevation rises or falls by 3.3% of that distance. Stating slope gradient as 33.3‰ means that for every kilometer of horizontal distance traveled, elevation rises or falls by 33.3 m (3.3% of that distance). A slope angle of 1.9° does not give us much of an idea how rise compares to run, but it can easily be converted to percentage by taking the tangent[2] of the angle in degrees:

$$\text{Gradient }(\%) = \tan(1.9) = 0.033 \times 100 = 3.3\%$$

Slope ratio of 1:30 means there's a 1 m vertical rise or fall for every 30 m run of horizontal distance. Slope ratio can also be expressed inversely as parts run to parts rise: 30:1. This means that for every 30 m run of horizontal distance, there is one 1 m vertical rise or fall. Dirt hogs should be familiar with the nomenclature of slope ratio as it's commonly found in many types of land surveys.

We can depict this slope geometrically as:

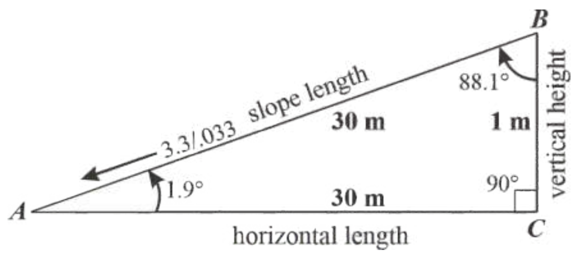


Figure 4. Right triangle with slope gradient 1:30. The grade may be expressed as percent or decimal; it is important to know which one is used. The arrow always points down the slope. The angle formed between the slope length and vertical height is found by the angle sum property of triangles: 180-(90+1.9).

Notice in **Figure 4** slope length and horizontal length are labeled the same. This would appear to violate the property of right triangles stating the hypotenuse is always the longest side. This rule is observed but the difference in length of the two sides is unapparent because length is rounded to the nearest meter. The exact slope length **S** can be found by the Pythagorean theorem:

$$\mathbf{S}^2 = \mathbf{H}^2 + \mathbf{V}^2$$

$$\mathbf{S} = \sqrt{\mathbf{H}^2 + \mathbf{V}^2}$$

$$\mathbf{S} = \sqrt{30^2 + 1^2} = \sqrt{901}$$

$$\mathbf{S} = 30.017\text{ m}$$

Carrying **S** out to three decimal places reveals a length .017 m (1.7 cm) longer than **H**, which suggests a relative difference of approximately .06%. For all practical purposes, if the reduction in horizontal length does not exceed 2 cm per full tape length (3000 cm), the reduction in horizontal length **H** can be ingored. This could be helpful in situations where high precision isn't required because slope length is easier to find than horizontal run. Gradients larger than 1:30 or angles larger than 2° should always be determined by finding the reduction to horizontal distance, a topic we examine further in Section 22.3.

[2] tan button on a scientific/engineering calculator.

Looking at **contour lines** on a map, the interval between two lines represents the change in elevation determined *as the horizontal distance between the lines*.

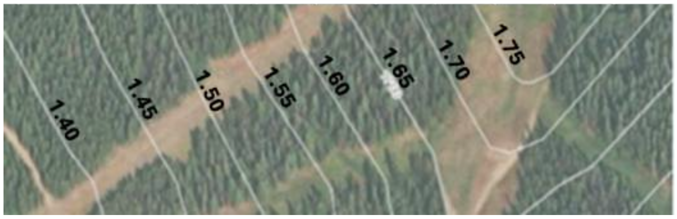


Figure 5. Topographic map with 5-unit contour lines (ft, m, etc.).

The map in **Figure 5** shows contour lines with 5-unit intervals therefore the difference in elevation, or vertical distance $\mathbf{V}$ between lines, is 5 units. To find slope gradient of the hillside, we need to find the horizontal distance $\mathbf{H}$ between the lines. We can measure the inclined slope length $\mathbf{S}$ using a tape measure, but we can't directly measure the horizontal distance between two points on a slope. Unless grade is smaller than 1:30, the value $\mathbf{H}$ must be found another way. In the next section, we'll describe one method to determine the horizontal and vertical distance between two distinct points on a slope and find its gradient.

22.3 Method for Computing Slope Gradient

Slope gradient can be measured precisely using a transit level instrument. In engineering and construction fields this is the gold standard. However, transit levels are bulky, expensive, and their set-up and movement time consuming. Slope gradient can also be measured using an inexpensive optical instrument called an Abney level or clinometer (**Figure 6**).

Figure 6. (**Left**) Transit level and rod. (**Right**) Clinometer or Abney level. The clinometer fits conveniently in any clothes pocket.

Although clinometers are not high precision instruments, they are easy to use and small enough to carry in the pocket. Accuracy is generally sufficient for classification purposes where a slope range is needed or to determine the direction of surface flow.

The clinometer is used to measure the slope angle denoted by $a°$ in **Figure 7**. The slope length $\mathbf{S}$ is measured with a tape. Surveyor's tapes that are graduated in 10ths are best for this purpose. Knowing the slope angle and slope length allows to compute $\mathbf{H}$ the true horizontal length using the formula:

$$\mathbf{H} = \mathbf{S}(\cos a°)$$

Where cos $a°$ is the cosine[3] of the slope angle measured by the clinometer.

[3] cos button on a scientific/engineering calculator.

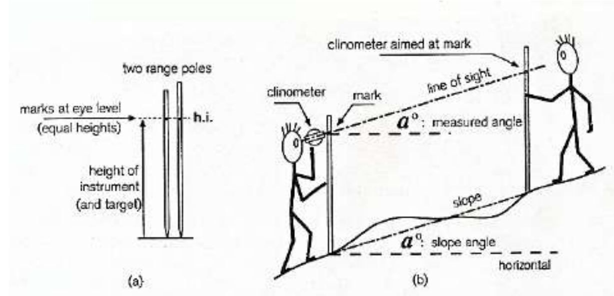


Figure 7. Determination of slope angle with a clinometer. (a) Two range poles of rods with equal height marks are used for marking a line sight parallel to the slope. (b) The measured angle of this line of sight is equal to the slope angle $a°$. *Image source: Loedeman 2000.*

The vertical rise $\mathbf{V}$ is computed using the Pythagorean relation $c^2=a^2+b^2$ for a right triangle. This relation can be restated using the notation:

$$\mathbf{S}^2 = \mathbf{H}^2 + \mathbf{V}^2$$

Where $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{V}$ are the measured slope length, horizontal length, and vertical rise corresponding to sides $\mathbf{c}$, $\mathbf{a}$, $\mathbf{b}$ in **Figure 2**.

Now solve algebraically for $\mathbf{V}$:

$$\mathbf{V}^2 = \mathbf{S}^2 - \mathbf{H}^2$$

$$\mathbf{V} = \sqrt{\mathbf{S}^2 - \mathbf{H}^2}$$

Once the vertical rise $\mathbf{V}$ has been determined, percent slope gradient is given by $\mathbf{V}/\mathbf{H}$ x 100.

In summary: to compute the vertical height $\mathbf{V}$ (change in elevation) between two points on a slope, the horizontal distance between the points must be determined. The horizontal distance will always be less than the slope length according to the Pythagorean theorem. The slope length $\mathbf{S}$ must be reduced to a horizontal length $\mathbf{H}$ by measuring the slope angle $a°$ as demonstrated by **Figure 8**. The difference $\mathbf{H}-\mathbf{S}$ yields the reduction from slope length to equivalent horizontal length. **Table 1** shows gradients, their equivalent slope angles and reduction of a fixed slope distance to horizontal length.

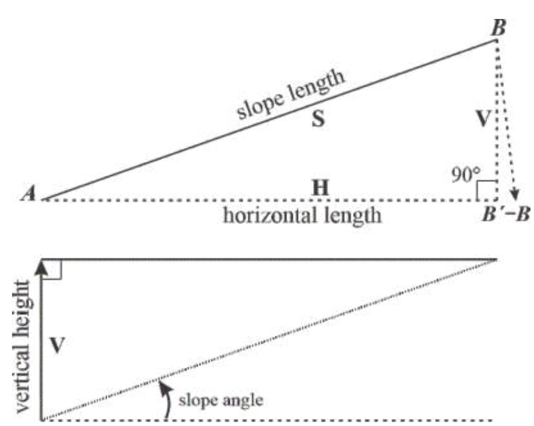


Figure 8. Slope length $\mathbf{S}$ must be reduced to a horizontal length $\mathbf{H}$ by taking account of the slope angle. The difference $\mathbf{B'}-\mathbf{B}$ between slope length $\mathbf{A}$-$\mathbf{B}$ and the equivalent horizontal length $\mathbf{A}$-$\mathbf{B'}$ depends on the size of the slope angle (gradient). Reduction can be ignored when slope angle is less than 2° in most circumstances.

Table 1
Values for slope length, vertical height, and horizontal distance and their equivalent expressions of slope gradient for a fixed 30 m (~100') tape measure. For example, a 2% slope would have 24-inch (60 cm) contours 100' apart, or 12 inches (30 cm) every 50' apart.

V (cm)	H (cm)	S (cm)	slope angle	slope percent	slope grade
30	2999.8	3000	0.6	1.0	1:100
60	2999.4	3000	1.1	2.0	1:50
75	2999.1	3000	1.4	2.5	1:40
100	2998.3	3000	1.9	3.3	1:30
150	2996.2	3000	2.9	5.0	1:20
200	2993.3	3000	3.8	6.7	1:15
300	2985.0	3000	5.7	10.1	1:10
600	2939.4	3000	11.5	20.4	1:5
750	2904.7	3000	14.5	25.8	1:4
1000	2828.4	3000	19.5	35.4	1:3
1500	2598.1	3000	30.0	57.7	1:2
2120	2122.6	3000	45.0	100	1:1

For long slopes it may be impractical to sight over the entire distance. For such a condition $\mathbf{H}$ and $\mathbf{V}$ may be measured by dividing the distance into parts. The average slope over the distance is the sum of the parts (**Figure 9**).

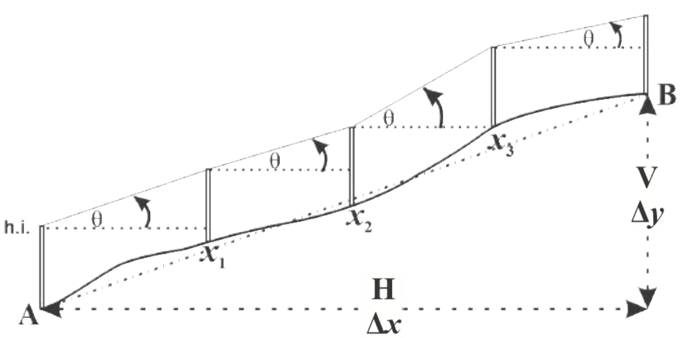
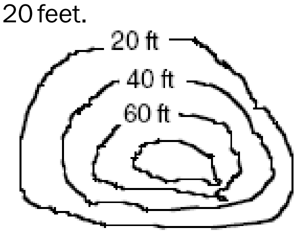


Figure 9. Sketch profile showing the methodology for determining average and piecewise slope gradient over a long distance. The vertical distance $\mathbf{V}$ is the sum of the number of piecewise sighting heights x_1, x_2, and x_3.

Core Intelligence

Aspect: Compass bearing that a slope faces measured clockwise from 0 to 360 degrees where 0 degrees is north-facing, 90 is east-facing, 180 is south-facing, and 270 is west-facing.

Contour lines: Lines connecting points of equal elevation above a reference surface (usually mean sea level) on a topographic map. The *contour interval* is the vertical distance or difference in elevation between contour lines. For example. in the graphic below the vertical distance between lines is 40 feet − 20 feet = 20 feet.



Contour lines provide two key pieces of information: how high we are (our elevation); and, steepness (gradient) of a slope.

Elevation: The vertical distance (height) of a point above or below a reference surface or datum. Elevations on a map may be measured relative to any reference point. The usual reference surface for topographic maps is mean sea level (msl): all elevations on the map are measured height above or below msl. In some survey work, for example land leveling, relative elevation is all that's needed for machine grading. The height of points on a grading map are scaled to neighboring points on the map that are, in turn, tied to a master benchmark elevation.

Gradient: The rate of change in value of some physical quantity over distance or time. In engineering and construction fields gradient is the ratio of the vertical distance (rise) and horizontal distance (run) indicating the inclination or steepness of a surface in a given direction.

Grade: The degree or amount of incline or decline of a surface, whether land, road, or railway. Grade is typically expressed as a percentage or ratio.

Pythagorean theorem: In Euclidean geometry, the Pythagorean theorem states a fundamental relationship between three sides of a right triangle:

$$c^2 = a^2 + b^2$$

where a and b are the opposite and adjacent sides, and c is the hypotenuse (see **Figure 2**).The Pythagorean theorem states that "the square of the hypotenuse is equal to the sum of the squares of the other two sides". This famous equation, which has numerous mathematical proofs, is widely employed in solving problems involving slope, distance, and area.

Slope gradient: In mathematics, *slope* or *gradient* is a number that indicates the steepness and direction of a line. The term *slope gradient* refers specifically to the rise or fall in surface elevation over a given distance or direction, rather than some other physical quantity like temperature or pressure.

Further Reading

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